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Page 1178-1197

THIRD-ORDER SANDWICH RESULTS FOR ANALYTIC UNIVALENT FUNCTIONS DEFINED BY INTEGRAL OPERATOR

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Abstract
In the present paper, we obtain Third-Order sandwich theorems for univalent functions by using some results of differential subordination and superordination for univalent functions involving integral operator.
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1-Introduction

Let $H(U)$ be the class of functions which are analytic in the open unit disk $U = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$. For $n \in N = \{1,2,3,\cdots\}\ and\ a \in \mathbb{C}$. Let $H[a,n] = \{f: f \in H(U)\ and\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$ and suppose that $H_0 = H[0,1]$. We denote by $\mathcal{A} \subset H(U)$ the class of all normalized analytic functions in $U$of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n\ ,\ (z \in U). \tag{1.1}$$

Let f and g be members of the analytic function class $H(U)$, f is said to be subordinate to g, or g is superordinate to f, if there exists a Schwarz function $h(z)$, analytic in U with $h(0) = 0\ and\ |h(z)| < 1\ , (z \in U)$,
such that $f(z) = g\big(h(z)\big), (z \in U)$. In such a case, we write $f \prec g$ or $f(z) \prec g(z), (z \in U)$. Furthermore, if the function g is univalent in U, then, we have the following equivalence: (see[3,17]).

$$f(z) \prec g(z) \ \leftrightarrow f(0) = g(0)\ and\ f(U) \subset g(U).$$

Jung, Kim and Srivastava [18] introduced the following integral operator:

$$\mathcal{Q}_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt\ , \tag{1.2}$$

where $(\alpha > 0, \beta > -1\ ; f \in \mathcal{A})$.
They showed that

$$\mathcal{Q}_\beta^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)\,\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+n)\Gamma(\beta+1)} a_n\, z^n. \tag{1.3}$$

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

It is easily verified from definition (1.3) that

$$z\left(\mathcal{Q}_{\beta}^{\alpha}f(z)\right)' = (\alpha+\beta)\mathcal{Q}_{\beta}^{\alpha-1}f(z) - (\alpha+\beta-1)\mathcal{Q}_{\beta}^{\alpha}f(z). \qquad (1.4)$$

The notion of the third-order differention subordination can be found in the work of Ponnusamy and Juneja [19]. The recent work by Tang et al. (see for example, [22] and [23]; see also [13]) The second and third-order differential subordination attracted many researchers in this field. For example, see ([1,2,4,5,6,7,8,9,10,11,12,13,14,15,16,20,21]).

Here, we discuss suitable class of admissible function associated with the integral operator and derive sufficient conditions on the normalized analytic function the sandwich condition.

2- Preliminaries

we need the following definition and lemmas to prove our results.

Definition (2.1) [3]. Let $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and the function $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the following third-order differential subordination:

$$\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \qquad (2.1)$$

then $p(z)$ is called a solution of the differential subordination (2.1). A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2.1), or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2.1) is said to be the best dominant.

Definition (2.2) [3]. Let $\mathbb{Q}$ denote the set of all functions q that are analytic and univalent on the set $\overline{U}\backslash E(q)$, where

$$E(q) = \{\xi: \xi \in \partial U: \lim_{z\to\xi} q(z) = \infty\},$$

and $min|q'(\xi)| = p > 0$ for $\xi \in \partial U\backslash E(q)$. Further, let the subclass of $\mathbb{Q}$ for which $q(0) = a$ be denoted by $\mathbb{Q}(a)$, with $\mathbb{Q}(0) = \mathbb{Q}_0$ and $\mathbb{Q}(1) = \mathbb{Q}_1 = \{q \in \mathbb{Q}: q(0) = 1\}$.

The subordination methodology is applied to an appropriate classes of admissible functions.

The following class of admissible functions is given by Antonino and Miller [17].

Definition (2.3) [3]. Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}$ and $n \in N|\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\Phi(r, s, t, u; z) \notin \Omega,$$

whenever

$$r = q(\xi)\,, s = k\xi q'(\xi), \qquad Re(\frac{t}{s}+1) \geq kRe(\frac{\xi q''(\xi)}{q'(\xi)}+1)$$

and

$$Re\left(\frac{u}{s}\right) \geq k^2 Re\left(\frac{\xi^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U\backslash E(q)$, and $k \geq n$.

Lemma (2.1) [3]. Let $p \in H[a, n]$ with $n \geq 2$. Also , let $q \in \mathbb{Q}(a)$ and satisfy the following conditions:

$$Re(\frac{\xi q''(\xi)}{q'(\xi)}) \geq 0, \text{ and } \left|\frac{zp'(Z)}{q'(\xi)}\right| \leq k,$$

where $z \in U, \xi \in \partial U \backslash E(q)$, and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\Phi \in \Psi_n[\Omega, q]$ and

$$\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then $p(z) \prec q(z), (z \in U)$.

Definition (2.4) [23]. Let $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and the function $h(z)$ be analytic in U. If the function $p(z)$ and $\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in U and satisfy the following Third-Order differential superordination:

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \qquad (2.2)$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination, or more simply, a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (2.2). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (2.2) is said to be the best subordinant.

Definition (2.5) [23]. Let Ω be a set in $\mathbb{C}$, $q \in H[a, n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi_n'[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\Phi(r, s, t, u; \xi) \in \Omega,$$

whenever

$$r = q(z), \qquad s = \frac{zq'(z)}{m}, \qquad Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2} Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$ and $m \geq n \geq 2$.

Lemma(2.2)[23]. Let $q \in H[a, n]$ and $\Phi \in \Psi_n'[\Omega, q]$. If $\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is univalent in U and $p \in \mathbb{Q}(a)$ satisfy the following conditions:

$$Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \qquad \text{and } \left|\frac{\xi p'(z)}{q'(z)}\right| \leq m,$$

where $z \in U, \xi \in \partial U$ and $m \geq n \geq 2$,

then

$$\Omega \subset \{\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\}$$

implies that

$$q(z) \prec p(z), (z \in U).$$

3- Third-Order Differential Subordination Results

Here, we introduce some differential subordination results by using the integral operator.

Definition (3.1). Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap H_0$. The class $\mathcal{L}_j[\Omega, q]$ of admissible functions consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; z) \notin \Omega,$$

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

whenever

$$a = q(\xi), b = \frac{\xi k q'(\xi) + (\alpha+\beta-1)q(\xi)}{\alpha+\beta},$$

$Re\left(\frac{(\alpha+\beta)^2 c-(\alpha+\beta-1)[2(\alpha+\beta)(a+b)-a]}{(\alpha+\beta)b-(\alpha+\beta-1)a}\right) \geq kRe\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right),$

and

$Re\left(\frac{(\alpha+\beta)^3(d-3c)+\left[(\alpha+\beta)^2(7(\alpha+\beta)-4)+(\alpha+\beta)\right]b+(\alpha+\beta)(\alpha+\beta-1)[5-9(\alpha+\beta)]a}{(\alpha+\beta)b-(\alpha+\beta-1)a}\right) \geq k^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$

where $z \in U, \xi \in \partial U/E(q)$ and $k \geq 2$.

Theorem (3.1). Let $\Phi \in \mathcal{L}_j[\Omega, q]$. If the functions $f \in \mathcal{A}$ and $q \in \mathbb{Q}_0 \cap H_0$ satisfy the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \ \left|\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}\right| \leq k \tag{3.1}$$

and

$$\{\Phi\left(\mathcal{Q}_\beta^{\alpha} f(z), \mathcal{Q}_\beta^{\alpha-1} f(z), \mathcal{Q}_\beta^{\alpha-2} f(z), \mathcal{Q}_\beta^{\alpha-3} f(z); z\right) : z \in U\} \subset \Omega\,, \tag{3.2}$$

then

$$\mathcal{Q}_\beta^{\alpha} f(z) \prec q(z), \qquad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = \mathcal{Q}_\beta^{\alpha} f(z). \tag{3.3}$$

From equation (1.4) and (3.3), we have

$$\mathcal{Q}_\beta^{\alpha-1} f(z) = \frac{zp'(z) + (\alpha+\beta-1)p(z)}{\alpha+\beta}. \tag{3.4}$$

By a similar argument, we get

$\mathcal{Q}_\beta^{\alpha-2} f(z) =$

$$\frac{z^2 p''(z) + [2(\alpha+\beta)-1]zp'(z) + (\alpha+\beta-1)^2 p(z)}{(\alpha+\beta)^2}, \tag{3.5}$$

and

$$\mathcal{Q}_\beta^{\alpha-3} f(z) = \frac{z^3 p'''(z) + 3(\alpha+\beta) z^2 p''(z) + \left[(\alpha+\beta)\left((\alpha+\beta)+1\right)\right] zp'(z) + (\alpha+\beta-1)^3 p(z)}{(\alpha+\beta)^3} \tag{3.6}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u) = r, b(r,s,t,u) = \frac{s + (\alpha+\beta-1)r}{\alpha+\beta},$$

$$c(r,s,t,u) = \frac{t + [2(\alpha+\beta)-1]s + (\alpha+\beta-1)^2 r}{(\alpha+\beta)^2} \tag{3.7}$$

and

$$d(r,s,t,u) = \frac{u + 3(\alpha+\beta)t + \left[(\alpha+\beta)\left((\alpha+\beta)+1\right)\right]s + (\alpha+\beta-1)^3 r}{(\alpha+\beta)^3}. \tag{3.8}$$

Let $\varphi(r,s,t,u) = \Phi(a,b,c,d) =$

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

$$\Phi\left(\begin{array}{c} r,\dfrac{s+(\alpha+\beta-1)r}{\alpha+\beta},\dfrac{t+[2(\alpha+\beta)-1]s+(\alpha+\beta-1)^2r}{(\alpha+\beta)^2}, \\ \dfrac{u+3(\alpha+\beta)t+\left[(\alpha+\beta)\left((\alpha+\beta)+1\right)\right]s+(\alpha+\beta-1)^3r}{(\alpha+\beta)^3}\end{array}\right). \tag{3.9}$$

The proof will make use of Lemma (2.1). Using equations (3.2) to (3.5), and from (3.8), we have

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z) = \Phi\left(\mathcal{Q}_\beta^\alpha f(z),\mathcal{Q}_\beta^{\alpha-1}f(z),\mathcal{Q}_\beta^{\alpha-2}f(z),\mathcal{Q}_\beta^{\alpha-3}f(z);z\right). \tag{3.10}$$

Hence, clearly, (3.2) leads to

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z) \in \Omega,$$

we note that

$$\frac{t}{s}+1=\frac{(\alpha+\beta)^2c-(\alpha+\beta-1)[2(\alpha+\beta)(a+b)-a]}{(\alpha+\beta)b-(\alpha+\beta-1)a}$$

and

$$\frac{u}{s}=\frac{(\alpha+\beta)^3(d-3c)+\left[(\alpha+\beta)^2(7(\alpha+\beta)-4)+(\alpha+\beta)\right]b+(\alpha+\beta)(\alpha+\beta-1)[5-9(\alpha+\beta)]a}{(\alpha+\beta)b-(\alpha+\beta-1)a}.$$

Thus clearly, the admissibility condition for $\Phi \in \mathcal{L}_j[\Omega,q]$ in Definition (3.1) is equivalent to admissibility condition for $\varphi \in \Psi_2[\Omega,q]$ as given in Definition (2.3) with $n=2$. Therefore, by using (3.1) and Lemma (2.1), we have

$$\mathcal{Q}_\beta^\alpha f(z) \prec q(z).$$

This completes the proof of theorem (3.1).

Our next results is a consequence of Theorem (3.1) for the case when the behavior of $q(z)$ on ∂U is not known.

Corollary (3.1). Let $\Omega \subset \mathbb{C}$ and the function q be univalent in U *with* $q(0)=1$. Let $\Phi \in \mathcal{L}_j[\Omega,q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z)=q(\rho z)$. If the function $f \in \mathcal{A}$ and q_ρ satisfies the following conditions:

$Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0$, $\left|\frac{\mathcal{Q}_\beta^{\alpha-1}f(z)}{q_\rho'(\xi)}\right| \leq k$, ($z \in U, \xi \in \partial U\backslash E(q_\rho)$ and $k \geq 2$.)

and

$$\Phi\left(\mathcal{Q}_\beta^\alpha f(z),\mathcal{Q}_\beta^{\alpha-1}f(z),\mathcal{Q}_\beta^{\alpha-2}f(z),\mathcal{Q}_\beta^{\alpha-3}f(z);z\right) \in \Omega,$$

then

$$\mathcal{Q}_\beta^\alpha f(z) \prec q_\rho(z), (z \in U).$$

Proof. By applying Theorem (3.1), we get

$$\mathcal{Q}_\beta^\alpha f(z) \prec q_\rho(z), (z \in U).$$

The result asserted by Corollary (3.1) is now deduced from following subordination property $q_\rho(z) \prec q(z)$, ($z \in U$).

This completes the proof of Corollary (3.1).

In particular case when $h(z)$ conformal transformation of U onto Ω such that $\Omega \neq \mathbb{C}$ is simply connected domain, thus $\Omega = h(z)$, we denote the class $\mathcal{L}_j[\mathrm{h(U),q}]$ by $\mathcal{L}_j[\mathrm{h,q}]$.
The following two results are immediate consequence of Theorem (3.1) and corollary (3.1).
Theorem (3.2). Let $\Phi \in \mathcal{L}_j[\mathrm{h,q}]$. If the following $f \in \mathcal{A}$ and $q \in \mathbb{Q}_0 \cap H_0$ satisfy the following conditions:

$$Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0\,, \left|\frac{\mathcal{Q}_\beta^{\alpha-1}f(z)}{q_\rho'(\xi)}\right| \leq k \qquad (3.11)$$

and

$$\Phi\left(\mathcal{Q}_\beta^{\alpha}f(z), \mathcal{Q}_\beta^{\alpha-1}f(z), \mathcal{Q}_\beta^{\alpha-2}f(z), \mathcal{Q}_\beta^{\alpha-3}f(z); z\right) \prec h(z). \qquad (3.12)$$

Then

$$\mathcal{Q}_\beta^{\alpha-1}f(z) \prec q(z)\,, (z \in U).$$

The next result is an immediate consequence of corollary (3.1).
Corollary (3.2). Let $\Omega \subset \mathbb{C}$ and q be univalent function in U with $q(0) = 1$. Also let $\Phi \in \mathcal{L}_j[\Omega, q]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}$ and q_ρ satisfies the following conditions:

$$Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0\,, \left|\frac{\mathcal{Q}_\beta^{\alpha-1}f(z)}{q_\rho'(\xi)}\right| \leq k\,, (z \in U, \xi \in \partial U/E(q_\rho) \text{ and } k \geq 2)$$

and

$$\Phi\left(\mathcal{Q}_\beta^{\alpha}f(z), \mathcal{Q}_\beta^{\alpha-1}f(z), \mathcal{Q}_\beta^{\alpha-2}f(z), \mathcal{Q}_\beta^{\alpha-3}f(z); z\right) \prec h(z),$$

then

$$\mathcal{Q}_\beta^{\alpha-1}f(z) \prec f(z), \qquad (z \in U).$$

The following result yield the best dominant of differential subordination (3.12).
Theorem (3.3). Let h be univalent function in U. Also let $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and φ be given by (3.10). Suppose that following differential equation:

$$\varphi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z) \qquad (3.13)$$

has a solution $q(z)$ with $q(0) = 1$, which satisfies the condition (3.1). If $f \in \mathcal{A}$ satisfies the condition (3.12) and if

$$\Phi\left(\mathcal{Q}_\beta^{\alpha}f(z), \mathcal{Q}_\beta^{\alpha-1}f(z), \mathcal{Q}_\beta^{\alpha-2}f(z), \mathcal{Q}_\beta^{\alpha-3}f(z); z\right),$$

is analytic in U, then

$$\mathcal{Q}_\beta^{\alpha}f(z) \prec q(z), \qquad (z \in U)$$

and $q(z)$ is the best dominant.
Proof. From Theorem (3.1), we see that q is a dominant of (3.12). Since q satisfies (3.13), it is also a solution of (3.12). Therefore, q will be dominated by all dominants. Hence q is the best dominant. This completes the proof of Theorem (3.3).

In view of Definition (3.1), and in special case when $q(z) = Mz$ ($M > 0$) ,the class $\mathcal{L}_j[\Omega, q]$ of admissible functions, denoted by $\mathcal{L}_j[\Omega, M]$ is expressed as follows.

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

Definition (3.2). Let Ω be set in ℂ and M > 0. The class $\mathcal{L}_j[\Omega, M]$ of admissible functions consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\Phi\left(\begin{array}{c} Me^{i\theta}, \frac{(k+(\alpha+\beta-1))Me^{i\theta}}{\alpha+\beta}, \frac{L+[[2(\alpha+\beta)-1]k+(\alpha+\beta-1)^2]Me^{i\theta}}{(\alpha+\beta)^2}, \\ \frac{N+3(\alpha+\beta)L+[\left((\alpha+\beta)\left((\alpha+\beta)+1\right)\right)k+(\alpha+\beta-1)^3]Me^{i\theta}}{(\alpha+\beta)^3}; z \end{array}\right) \notin \Omega, \quad (3.14)$$

whenever $z \in U$,

$$Re\left(Le^{-i\theta}\right) \geq (k-1)kM$$

and

$$Re\left(Ne^{-i\theta}\right) \geq 0, \qquad \forall\theta \in \mathbb{R}, k \geq 2.$$

Corollary (3.3). Let $\Phi \in \mathcal{L}_j[\Omega, M]$. If the function $f \in \mathcal{A}$ satisfies the following conditions:

$$\left|Q_\beta^{\alpha-1} f(z)\right| \leq kM, \qquad (z \in \mathrm{U}, k \geq 2; M > 0)$$

and

$$\Phi\left(Q_\beta^\alpha f(z), Q_\beta^{\alpha-1} f(z), Q_\beta^{\alpha-2} f(z), Q_\beta^{\alpha-3} f(z); z\right) \in \Omega,$$

then

$$\left|Q_\beta^\alpha f(z)\right| < M.$$

In the special case, when $\Omega = q(U) = \{w: |w| < M\}$, the class $\mathcal{L}_j[\Omega, q]$ is simply denoted by $\mathcal{L}_j[M]$. Corollary (3.3) can now be rewritten in the following form,

Corollary (3.4). Let $\Phi \in \mathcal{L}_j[M]$. If the function $f \in \mathcal{A}$ satisfies the following conditions:

$$\left|Q_\beta^{\alpha-1} f(z)\right| \leq kM, \qquad (z \in \mathrm{U}, k \geq 2; M > 0)$$

and

$$\left|\left(Q_\beta^\alpha f(z), Q_\beta^{\alpha-1} f(z), Q_\beta^{\alpha-2} f(z), Q_\beta^{\alpha-3} f(z); z\right)\right| < M,$$

then

$$\left|Q_\beta^\alpha f(z)\right| < M.$$

Corollary (3.5). Let $k \geq 2$, *and* $M > 0$. If the function $f \in \mathcal{A}$ satisfies the following conditions:

$$\left|Q_\beta^{\alpha-1} f(z)\right| \leq kM$$

and

$$\left|Q_\beta^{\alpha-1} f(z) - Q_\beta^\alpha f(z)\right| \leq \frac{M}{\alpha+\beta},$$

then

$$\left|Q_\beta^\alpha f(z)\right| \leq M.$$

Proof. Let $\Phi(a, b, c, d; z) = b - a, \Omega = h(U)$, where $h(z) = \frac{(k-1)Mz}{\alpha+\beta}, z \in U, M > 0$.

By using Corollary (3.3), we need to show that $\Phi \in \mathcal{L}_j[\Omega, M]$, That is the admissibility condition (3.14) is satisfied. This follows readily, since it is seen that

$$|\Phi(a, b, c, d; z)| = \left|\frac{(k-1)}{\alpha+\beta} Me^{i\theta}\right| = \frac{k-1}{\alpha+\beta} M \geq \frac{M}{\alpha+\beta},$$

whenever $z \in U, \theta \in \mathbb{R}$ and $k \geq 2$.

Definition (3.3). Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_1 \cap H_1$. The class $\mathcal{L}_{j,1}[\Omega, q]$ of admissible functions consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = q(\xi), \qquad b = \frac{k\xi q'(\xi) + (\alpha+\beta)q(\xi)}{\alpha+\beta},$$

$$Re\left(\frac{(\alpha+\beta)[c+a-2b]}{b-a}\right) \geq kRe\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

and

$$Re\left(\frac{(\alpha+\beta)^2[d-3c+3b-a]-(\alpha+\beta)[5a+3(c-2b)]}{b-a} + 2\right) \geq k^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq 2$.

Theorem (3.4). Let $\Phi \in \mathcal{L}_{j,1}[\Omega, q]$. If the functions $f \in \mathcal{A}$ and $q \in \mathbb{Q}_1 \cap H_1$ satisfy the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{zq'(z)}\right| \leq k \tag{3.15}$$

and

$$\left\{\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right) : z \in U\right\} \subset \Omega, \tag{3.16}$$

then

$$\frac{\mathcal{Q}_\beta^{\alpha} f(z)}{z} \prec q(z), \qquad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = \frac{\mathcal{Q}_\beta^{\alpha} f(z)}{z}. \tag{3.17}$$

From equation (1.4) and (3.17), we have

$$\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z} = \frac{zp'(z) + (\alpha+\beta)p(z)}{\alpha+\beta}. \tag{3.18}$$

By a similar argument, we get

$$\frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z} = \frac{z^2 p''(z) + [2(\alpha+\beta)+1]zp'(z) + (\alpha+\beta)^2 p(z)}{(\alpha+\beta)^2} \tag{3.19}$$

and

$$\frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z} = \frac{z^3 p'''(z) + 3(\alpha+\beta+1)z^2 p''(z) + [3(\alpha+\beta)(\alpha+\beta+1)]zp'(z) + (\alpha+\beta)^3 p(z)}{(\alpha+\beta)^3}. \tag{3.20}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

$$a(r,s,t,u)=r\ ,b(r,s,t,u)=\frac{s+(\alpha+\beta)r}{\alpha+\beta},$$
$$c(r,s,t,u)=\frac{t+[2(\alpha+\beta)+1]+(\alpha+\beta)^2r}{(\alpha+\beta)^2} \tag{3.21}$$

and

$$d(r,s,t,u)=\frac{u+3(\alpha+\beta+1)t+[3(\alpha+\beta)(\alpha+\beta+1)+1]s+(\alpha+\beta)^3r}{(\alpha+\beta)^3}. \tag{3.22}$$

Let

$$\varphi(r,s,t,u)=\Phi(a,b,c,d;z)=$$
$$\Phi\left(\begin{array}{c} r,\frac{s+(\alpha+\beta)r}{\alpha+\beta},\frac{t+[2(\alpha+\beta)+1]+(\alpha+\beta)^2r}{(\alpha+\beta)^2}, \\ \frac{u+3(\alpha+\beta+1)t+[3(\alpha+\beta)(\alpha+\beta+1)+1]s+(\alpha+\beta)^3r}{(\alpha+\beta)^3} \end{array};z\right). \tag{3.23}$$

The proof will make use of Lemma (2.1). Using the equations (3.17) to (3.20), and from (3.23), we have

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=$$
$$\Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z},\frac{\mathcal{Q}_\beta^{\alpha-1}f(z)}{z},\frac{\mathcal{Q}_\beta^{\alpha-2}f(z)}{z},\frac{\mathcal{Q}_\beta^{\alpha-3}f(z)}{z};z\right). \tag{3.24}$$

Hence, clearly, (3.16) becomes

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega,$$

we note that

$$\frac{t}{s}+1=\frac{(\alpha+\beta)[c+a-2b]}{b-a}$$

and

$$\frac{u}{s}=\frac{(\alpha+\beta)^2[d-3c+3b-a]-(\alpha+\beta)[5a+3(c-2b)]}{b-a}+2.$$

Thus clearly, the admissibility condition for $\Phi\in\mathcal{L}_{j,1}[\Omega,q]$ in Definition (3.3) is equivalent to admissibility condition for $\varphi\in\Psi_2[\Omega,q]$ as given in Definition (2.3) with $n=2$. Therefore, by using (3.13) and Lemma (2.1), we have

$$\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}\prec q(z).$$

This completes the proof of theorem (3.4).

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\mathcal{L}_{j,1}[\mathrm{h(U)},q]$ is written as $\mathcal{L}_{j,1}[\Omega,q]$. This leads to the following immediate consequence of Theorem (3.4) is sated below.

Theorem (3.5). Let $\Phi\in\mathcal{L}_{j,1}[\Omega,q]$. If the functions $f\in\mathcal{A}$ and $q\in\mathbb{Q}_1$ satisfy the following conditions:

$$Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right)\geq 0,\qquad \left|\frac{\mathcal{Q}_\beta^{\alpha-1}f(z)}{zq_\rho'(\xi)}\right|\leq k, \tag{3.25}$$

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

and

$$\Phi\left(\frac{\mathcal{Q}_{\beta}^{\alpha} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-3} f(z)}{z}; z\right) \prec h(z) \qquad (3.26)$$

then

$$\frac{\mathcal{Q}_{\beta}^{\alpha} f(z)}{z} \prec q(z), \qquad (z \in U).$$

In view of Definition (3.3) and in the special case $q(z) = Mz$, $M > 0$, the class of admissible functions $\mathcal{L}_{j,1}[\Omega, q]$, denoted by $\mathcal{L}_{j,1}[\Omega, M]$ is expressed as follows.

Definition (3.4). Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class $\mathcal{L}_{j,1}[\Omega, q]$ of admissible functions consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that:

$$\Phi\left(\begin{array}{c} Me^{i\theta}, \frac{k+(\alpha+\beta)}{\alpha+\beta} Me^{i\theta}, \frac{L+([2(\alpha+\beta)+1]k+(\alpha+\beta)^2)Me^{i\theta}}{(\alpha+\beta)^2}, \\ \frac{N+3(\alpha+\beta+1)L+([3(\alpha+\beta)(\alpha+\beta+1)+1]k+(\alpha+\beta)^3)Me^{i\theta}}{(\alpha+\beta)^3} \end{array}; z\right) \notin \Omega \qquad (3.27)$$

whenever

$$z \in U, \qquad Re(Le^{-i}\ ) \geq (k-1)kM$$

and

$$Re\left(Ne^{-i}\ \right) \geq 0, \qquad \forall \theta \in \mathbb{R}; k \geq 2.$$

Corollary (3.6). Let $\Phi \in \mathcal{L}_{j,1}[\Omega, q]$. If the function $f \in \mathcal{A}$ satisfy the following conditions:

$$\left|\frac{\mathcal{Q}_{\beta}^{\alpha-1} f(z)}{z}\right| \leq kM, \qquad (z \in U, k \geq 2; M > 0),$$

and

$$\Phi\left(\frac{\mathcal{Q}_{\beta}^{\alpha} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-3} f(z)}{z}; z\right) \in \Omega,$$

then

$$\left|\frac{\mathcal{Q}_{\beta}^{\alpha-1} f(z)}{z}\right| < M.$$

In the special case, when $\Omega = q(U) = \{w: |w| < M\}$, the class $\mathcal{L}_{j,1}[\Omega, q]$ is simply denoted by $\mathcal{L}_{j,1}[M]$.Corollary (3.6) can now be rewritten in the following form.

Corollary (3.7). Let $\Phi \in \mathcal{L}_{j,1}[\Omega, q]$. If the function $f \in \mathcal{A}$ satisfy the following conditions:

$$\left|\frac{\mathcal{Q}_{\beta}^{\alpha-1} f(z)}{z}\right| \leq kM, \qquad (z \in U, k \geq 2; M > 0),$$

and

$$\left|\Phi\left(\frac{\mathcal{Q}_{\beta}^{\alpha} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_{\beta}^{\alpha-3} f(z)}{z}; z\right)\right| < M,$$

then

$$\left|\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}\right| < M.$$

Definition (3.5). Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_1 \cap H_1$. The class $\mathcal{L}_{j,2}[\Omega, q]$ of admissible functions consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = q(\xi), \qquad b = \frac{1}{\alpha+\beta}\left[\frac{k\xi q'(\xi) + (\alpha+\beta)\big(q(\xi)\big)^2}{q(\xi)}\right],$$

$$Re\left(\frac{(\alpha+\beta)[cb + 2a^2 - 3ab]}{b-a}\right) \geq k\, Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right),$$

and

$$Re\Big(\big[bc(d-c)(\alpha+\beta)^2 - b(\alpha+\beta)(c-b)[(\alpha+\beta)(3a-b+1)+3]\big](a-b)^{-1} - 2a(\alpha+\beta) + (\alpha+\beta)(b-a)[(\alpha+\beta)(b-5a)-3] + 2\Big) \geq k^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U,\ \xi \in \partial U \backslash E(q)$ and $k \geq 2$.

Theorem (3.6). Let $\Phi \in \mathcal{L}_{j,2}[\Omega, q]$. If the functions $f \in \mathcal{A}$ and $q \in \mathbb{Q}_1 \cap H_1$ satisfy the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}\right| \leq k, \tag{3.28}$$

and

$$\left\{\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right) : z \in U\right\} \subset \Omega, \tag{3.29}$$

then

$$\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)} \prec q(z), \qquad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}. \tag{3.30}$$

From equation (1.4) and (3.30), we have

$$\frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)} = \frac{1}{\alpha+\beta}\left[\frac{zp'(z) + (\alpha+\beta)p^2(z)}{p(z)}\right] = \frac{A}{\alpha+\beta}. \tag{3.31}$$

By a similar argument, we get

$$\frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)} = \frac{B}{\alpha+\beta} \tag{3.32}$$

and

$$\frac{Q_{\beta}^{\alpha-4}f(z)}{Q_{\beta}^{\alpha-3}f(z)} = \frac{1}{\alpha+\beta}\left[B + B^{-1}(C + A^{-1}D - A^{-2}C^{2})\right], \tag{3.33}$$

where

$$B = \frac{zp'(z)}{p(z)} + (\alpha+\beta)p(z) + \frac{\frac{z^2p''(z)+zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2 + (\alpha+\beta)zp'(z)}{\frac{zp'(z)}{p(z)} + (\alpha+\beta)p(z)},$$

$$C = \frac{z^2p''(z)+zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2 + (\alpha+\beta)zp'(z)$$

and

$$D = \frac{z^3p'''(z)+3z^2p''(z)+zp'(z)}{p(z)} - \frac{3z^2\left(p'(z)\right)^2 + 3z^3p''(z)p'(z)}{\left(p(z)\right)^2} + 2\left(\frac{zp'(z)}{p(z)}\right)^3$$
$$+ (\alpha+\beta)z^2p''(z) + (\alpha+\beta)zp'(z).$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u) = r, b(r,s,t,u) = \frac{1}{\alpha+\beta}\left[\frac{s+(\alpha+\beta)r^2}{r}\right] = \frac{E}{\alpha+\beta},$$

$c(r,s,t,u) =$

$$\frac{1}{\alpha+\beta}\left[\frac{s+(\alpha+\beta)r^2}{r} + \frac{\frac{t+s}{r} - \left(\frac{s}{r}\right)^2 + (\alpha+\beta)s}{\frac{s}{r} + (\alpha+\beta)r}\right] = \frac{F}{\alpha+\beta} \tag{3.34}$$

and

$$d(r,s,t,u) = \frac{1}{\alpha+\beta}\left[F + F^{-1}(L + HE^{-1} - E^{-2}L^{2})\right], \tag{3.35}$$

where

$$L = \frac{t+s}{r} - \left(\frac{s}{r}\right)^2 + (\alpha+\beta)s$$

and

$$H = \frac{u+3t+s}{r} - 3\left(\frac{s}{r}\right)^2 - 3\frac{st}{r^2} + 2\left(\frac{s}{r}\right)^3 + (\alpha+\beta)(s+t).$$

Let

$\varphi(r,s,t,u) = \Phi(a,b,c,d) =$

$$\Phi\left(r, \frac{E}{\alpha+\beta}, \frac{F}{\alpha+\beta}, \frac{1}{\alpha+\beta}\left[F + F^{-1}(L + HE^{-1} - E^{-2}L^{2})\right]\right). \tag{3.36}$$

The proof will make use of Lemma (2.1). Using the equations (3.30) to (3.33), and from (3.36), we have

$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) =$

$$\Phi\left(\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}, \frac{Q_{\beta}^{\alpha-2}f(z)}{Q_{\beta}^{\alpha-1}f(z)}, \frac{Q_{\beta}^{\alpha-3}f(z)}{Q_{\beta}^{\alpha-2}f(z)}, \frac{Q_{\beta}^{\alpha-4}f(z)}{Q_{\beta}^{\alpha-3}f(z)}; z\right). \tag{3.37}$$

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

Hence, clearly (3.29) leads to

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

We note that

$$\frac{t}{s} + 1 = \frac{(\alpha+\beta)[cb + 2a^2 - 3ab]}{b-a}$$

and

$$\frac{u}{s} =$$

$\left[bc(d-c)(\alpha+\beta)^2 - b(\alpha+\beta)(c-b)[(\alpha+\beta)(3a-b+1)+3]\right](a-b)^{-1} - 2a(\alpha+\beta) + (\alpha+\beta)(b-a)[(\alpha+\beta)(b-5a)-3] + 2.$

Thus clearly, the admissibility condition for $\Phi \in \mathcal{L}_{j,2}[\Omega, q]$ in Definition (3.5) is equivalent to admissibility condition for $\varphi \in \Psi_2[\Omega, q]$ as given in Definition (2.3) with $n = 2$. Therefore, by using (3.30) and Lemma (2.1), we have

$$\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)} \prec q(z).$$

This completes the proof of theorem (3.6).

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\mathcal{L}_{j,2}[\mathrm{h(U)}, q]$ is written as $\mathcal{L}_{j,2}[\Omega, q]$. This leads to the following immediate consequence of Theorem (3.6) is sated below without proof.

Theorem (3.7). Let $\Phi \in \mathcal{L}_{j,2}[\Omega, q]$. If the functions $f \in \mathcal{A}$ and $q \in \mathbb{Q}_1$ satisfy the following conditions (3.29) and

$$\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right) \prec h(z),$$

then

$$\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)} \prec q(z), \qquad (z \in U).$$

4- Third-Order Differential Superordination Results

Definition (4.1). Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap H_0$ with $q'(z) \neq 0$. The class of admissible functions $\mathcal{L}_j'[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$,that satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = q(z), \qquad b = \frac{zq'(z) + m(\alpha+\beta-1)q(z)}{m(\alpha+\beta)},$$

$$Re\left(\frac{(\alpha+\beta)^2 c - (\alpha+\beta-1)[2(\alpha+\beta)(a+b)-a]}{(\alpha+\beta)b - (\alpha+\beta-1)a}\right) \leq \frac{1}{m} Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

$$Re\left(\frac{(\alpha+\beta)^3(d-3c)+[(\alpha+\beta)^2(7(\alpha+\beta)-4)+(\alpha+\beta)]b+(\alpha+\beta)(\alpha+\beta-1)[5-9(\alpha+\beta)]a}{(\alpha+\beta)b-(\alpha+\beta-1)a}\right)\leq\frac{1}{m^2}Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z\in U,\xi\in\partial U$ and $m\geq 2$.

Theorem (4.1). Let $\Phi\in\mathcal{L}_j'[\Omega,q]$. If the functions $f\in\mathcal{A}$, with $\mathcal{Q}_\beta^\alpha f(z)\in\mathbb{Q}_0$ and if $q\in H_0$ with $q'(z)\neq 0$, satisfying the following conditions:

$$Re\left(\frac{\xi q''(z)}{q'(z)}\right)\geq 0,\qquad\left|\frac{\mathcal{Q}_\beta^\alpha f(z)}{q'(z)}\right|\leq m\tag{4.1}$$

and the function

$$\Phi\left(\mathcal{Q}_\beta^\alpha f(z),\mathcal{Q}_\beta^{\alpha-1}f(z),\mathcal{Q}_\beta^{\alpha-2}f(z),\mathcal{Q}_\beta^{\alpha-3}f(z);z\right)$$

is univalent in U, then

$$\Omega\subset\left\{\Phi\left(\mathcal{Q}_\beta^\alpha f(z),\mathcal{Q}_\beta^{\alpha-1}f(z),\mathcal{Q}_\beta^{\alpha-2}f(z),\mathcal{Q}_\beta^{\alpha-3}f(z);z\right):z\in U\right\},\tag{4.2}$$

implies that

$$q(z)\prec\mathcal{Q}_\beta^\alpha f(z),\qquad(z\in U).$$

Proof. Let the function $p(z)$ be defined by (3.2) and φ by (3.8). Since $\Phi\in\mathcal{L}_j'[\Omega,q]$. From (3.10) and (4.2), we have

$$\Omega\subset\{\Phi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\}.$$

From (3.7) and (3.8), we see that the admissibility condition for $\Phi\in\mathcal{L}_j'[\Omega,q]$ in Definition (4.1) is equivalent to the admissibility for $\varphi\in\Psi_n'[\Omega,q]$ as given in Definition (2.5) with $n=2$. Hence $\varphi\in\Psi_2'[\Omega,q]$ and by using (4.2) and Lemma (2.2), we find that

$$q(z)\prec\mathcal{Q}_\beta^\alpha f(z),\qquad(z\in U).$$

This completes the proof of Theorem (4.1).

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\mathcal{L}_j'[\mathrm{h}(\mathrm{U}),q]$ is written as $\mathcal{L}_j'[\mathrm{h},q]$. This leads to the following immediate consequence of Theorem (4.1) is sated below.

Theorem (4.2). Let $\Phi\in\mathcal{L}_j'[\mathrm{h},q]$ and let h be analytic in U. If the function $f\in\mathcal{A}$, and $\mathcal{Q}_\beta^\alpha f(z)\in\mathbb{Q}_0$, and if $q\in H_0$ with $q'(z)\neq 0$, satisfying the following conditions (4.1) and the function

$$\Phi\left(\mathcal{Q}_\beta^\alpha f(z),\mathcal{Q}_\beta^{\alpha-1}f(z),\mathcal{Q}_\beta^{\alpha-2}f(z),\mathcal{Q}_\beta^{\alpha-3}f(z);z\right)$$

is univalent in U, then

$$h(z)\prec\Phi\left(\mathcal{Q}_\beta^\alpha f(z),\mathcal{Q}_\beta^{\alpha-1}f(z),\mathcal{Q}_\beta^{\alpha-2}f(z),\mathcal{Q}_\beta^{\alpha-3}f(z);z\right)\tag{4.3}$$

implies that

$$q(z)\prec\mathcal{Q}_\beta^\alpha f(z),\qquad(z\in U).$$

Theorem (4.1) and (4.2) can only be used to obtain subordination for the Third-Order differential superordination of the form (4.2) or (4.3). The following theorem gives the existence of the best subordinant of (4.3) for suitable Φ.

Theorem (4.3). Let the function h be univalent in U. Also let $\Phi:\mathbb{C}^4\times\overline{U}\to\mathbb{C}$ and φ be given by (3.9). Suppose that the following differential equation:

$$\varphi(q(z),zq'(z),z^2q''(z),z^3q'''(z);z)=h(z)\tag{4.4}$$

Advances in Mechanics
Volume 10, Issue 1, 2022
Page 1178-1197

has a solution $q(z) \in \mathbb{Q}_\circ$. If the functions $f \in \mathcal{A}$, and $\mathcal{Q}_\beta^\alpha f(z) \in \mathbb{Q}_0$, and if $q \in H_0$ with $q'(z) \neq 0$, satisfying the following conditions (4.1) and the function

$$\Phi\left(\mathcal{Q}_\beta^\alpha f(z), \mathcal{Q}_\beta^{\alpha-1} f(z), \mathcal{Q}_\beta^{\alpha-2} f(z), \mathcal{Q}_\beta^{\alpha-3} f(z); z\right)$$

is analytic in U, then

$$h(z) \prec \Phi\left(\mathcal{Q}_\beta^\alpha f(z), \mathcal{Q}_\beta^{\alpha-1} f(z), \mathcal{Q}_\beta^{\alpha-2} f(z), \mathcal{Q}_\beta^{\alpha-3} f(z); z\right)$$

implies that

$$q(z) \prec \mathcal{Q}_\beta^\alpha f(z), \qquad (z \in U)$$

and $q(z)$ is the best subordinant.

Proof. By applying Theorem (4.1) and (4.2), we deduce that q is a subordinant of (4.3). Since q satisfies (4.4), it is also a solution of (4.3) and therefore, q will be subordinant by all subordinants. Hence q is the best subordinant. This completes the proof of Theorem (4.3).

Definition (4.2). Let Ω be a set in $\mathbb{C}$ and $q \in H_1$ with $q'(z) \neq 0$. The class of admissible function $\mathcal{L}'_{j,1}[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$,that satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = q(z), \qquad b = \frac{zq'(z) + m(\alpha+\beta)q(z)}{m(\alpha+\beta)},$$

$$Re\left(\frac{(\alpha+\beta)[c+a-2b]}{b-a}\right) \leq \frac{1}{m} Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$Re\left(\frac{(\alpha+\beta)^2[d-3c+3b-a]-(\alpha+\beta)[5a+3(c-2b)]}{b-a} + 2\right) \leq \frac{1}{m^2} Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$ and $m \geq 2$.

Theorem (4.4). Let $\Phi \in \mathcal{L}'_{j,1}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z} \in \mathbb{Q}_1$, and if $q \in H_1$with $q'(z) \neq 0$, satisfying the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{zq'(z)}\right| \leq m \tag{4.5}$$

and the function

$$\Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right),$$

is univalent in U, then

$$\Omega \subset \left\{\Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right) : z \in U\right\} \tag{4.6}$$

implies that

$$q(z) \prec \frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \qquad (z \in U).$$

Proof. Let the function $p(z)$ be defined by (3.17) and φ by (3.23). Since $\Phi \in \mathcal{L}'_{j,1}[\Omega, q]$, we find from (3.24) and (4.6) that

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\}$$

From the equations (3.21) and (3.22), we see that the admissibility condition for $\Phi \in \mathcal{L}'_{j,1}[\Omega, q]$ in Definition (4.1) is equivalent to the admissibility condition for φ as given in Definition (2.3) with $n = 2$. Hence $\varphi \in \Psi'_2[\Omega, q]$ and by using (4.6) and Lemma (2.2), we have

$$q(z) \prec \frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \qquad (z \in U).$$

This completes the proof of Theorem (4.4).

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\mathcal{L}'_{j,1}[\mathrm{h}(\mathrm{U}), q]$ is written as $\mathcal{L}'_{j,1}[\mathrm{h}, q]$. This leads to the following immediate consequence of Theorem (4.4).

Theorem (4.5). Let $\Phi \in \mathcal{L}'_{j,1}[\mathrm{h}, q]$ and h be analytic in U. If the functions $f \in \mathcal{A}$, with $q \in H_1$ and $q'(z) \neq 0$, satisfying the following conditions (4.5) and the function

$$\Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right)$$

is univalent in U,then

$$h(z) \prec \Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right),$$

implies that

$$q(z) \prec \frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \qquad (z \in U).$$

Definition (4.3). Let Ω be a set in $\mathbb{C}$ and $q \in H_1$ with $q'(z) \neq 0$. The class of admissible functions $\mathcal{L}'_{j,2}[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$,that satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = q(z), \qquad b = \frac{1}{\alpha+\beta}\left[\frac{zq'(z) + m(\alpha+\beta)\big(q(z)\big)^2}{mq(z)}\right],$$

$$Re\left(\frac{(\alpha+\beta)[cb + 2a^2 - 3ab]}{b-a}\right) \leq \frac{1}{m} Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$Re\big(\big[bc(d-c)(\alpha+\beta)^2 - b(\alpha+\beta)(c-b)[(\alpha+\beta)(3a-b+1)+3]\big](a-b)^{-1} - 2a(\alpha+\beta) + (\alpha+\beta)(b-a)[(\alpha+\beta)(b-5a)-3]+2\big) \leq \frac{1}{m^2} Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U,\ \xi \in \partial U \backslash E(q)$ and $k \geq 2$.

Theorem (4.6). Let $\Phi \in \mathcal{L}'_{j,2}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)} \in \mathbb{Q}_1$, and if $q \in H_1$with $q'(z) \neq 0$, satisfying the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}\right| \leq m \tag{4.7}$$

and the function

$$\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right) : z \in U\right\} \tag{4.8}$$

implies that

$$q(z) \prec \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \qquad (z \in U).$$

Proof. Let the function $p(z)$ be defined by (3.30) and φ by (3.36). Since $\Phi \in \mathcal{L}'_{j,2}[\Omega, q]$, we find from (3.37) and (4.8) that

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in U\}.$$

From the equations (3.34) and (3.35), we see that the admissibility condition for $\Phi \in \mathcal{L}'_{j,2}[\Omega, q]$ in Definition (4.3) is equivalent to the admissibility condition for φ as given in Definition (2.5) when $n = 2$. Hence $\varphi \in \Psi'_2[\Omega, q]$ and by using (4.7) and Lemma (2.2), we have

$$q(z) \prec \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \qquad (z \in U).$$

This completes the proof of Theorem (4.6).

Theorem (4.7). Let $\Phi \in \mathcal{L}'_{j,2}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)} \in \mathbb{Q}_1$, and if $q \in H_1$with $q'(z) \neq 0$, satisfying the following conditions (4.7) and the function

$$\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right),$$

is univalent in U, then

$$h(z) \prec \Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right)$$

implies that

$$q(z) \prec \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^{\alpha} f(z)}, \qquad (z \in U).$$

5- Sandwich Results

By combining Theorem (3.2) and (4.2), we obtain the following sandwich-type theorem.

Theorem (5.1). Let h_1 and q_1 be analytic functions in U. Also let h_2 be univalent function in U and $q_2 \in \mathbb{Q}_0$ with $q_1(0) = q_2(0) = 1$ and $\Phi \in \mathcal{L}_j[h_2, q_2] \cap \mathcal{L}'_j[h_1, q_1]$. If the function $f \in \mathcal{A}$ with $\mathcal{Q}_\beta^\alpha f(z) \in \mathbb{Q}_0 \cap H_0$ and the function

$$\Phi\left(\mathcal{Q}_\beta^\alpha f(z), \mathcal{Q}_\beta^{\alpha-1} f(z), \mathcal{Q}_\beta^{\alpha-2} f(z), \mathcal{Q}_\beta^{\alpha-3} f(z); z\right),$$

is univalent in U, and if the conditions (3.1) and (4.1) are satisfied, then

$$h_1(z) \prec \Phi\left(\mathcal{Q}_\beta^\alpha f(z), \mathcal{Q}_\beta^{\alpha-1} f(z), \mathcal{Q}_\beta^{\alpha-2} f(z), \mathcal{Q}_\beta^{\alpha-3} f(z); z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \mathcal{Q}_\beta^\alpha f(z) \prec q_2(z), \qquad (z \in U). \tag{5.1}$$

If, on the order hand, we combine Theorem (3.5) and (4.5), we obtain the following sandwich-type Theorem.

Theorem (5.2). Let h_1 and q_1 be analytic functions in U. Also let h_2 be univalent function in U and $q_2 \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\Phi \in \mathcal{L}_{j,1}[h_2, q_2] \cap \mathcal{L}'_{j,1}[h_1, q_1]$. If the function $f \in \mathcal{A}$ with $\frac{\mathcal{Q}_\beta^\alpha f(z)}{z} \in \mathbb{Q}_1 \cap H_1$ and the function

$$\Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right),$$

is univalent in U, and if the conditions (3.15) and (4.5) are satisfied, then

$$h_1(z) \prec \Phi\left(\frac{\mathcal{Q}_\beta^\alpha f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{z}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{z}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{\mathcal{Q}_\beta^\alpha f(z)}{z} \prec q_2(z), \qquad (z \in U). \tag{5.2}$$

Theorem (5.3). Let h_1 and q_1 be analytic functions in U. Also let h_2 be univalent function in U and $q_2 \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\Phi \in \mathcal{L}_{j,2}[h_2, q_2] \cap \mathcal{L}'_{j,2}[h_1, q_1]$. If the function $f \in \mathcal{A}$ with $\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^\alpha f(z)} \in \mathbb{Q}_1 \cap H_1$ and the function

$$\Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^\alpha f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right),$$

is univalent in U, and if the conditions (3.28) and (4.7) are satisfied, then

$$h_1(z) \prec \Phi\left(\frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^\alpha f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-2} f(z)}{\mathcal{Q}_\beta^{\alpha-1} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-3} f(z)}{\mathcal{Q}_\beta^{\alpha-2} f(z)}, \frac{\mathcal{Q}_\beta^{\alpha-4} f(z)}{\mathcal{Q}_\beta^{\alpha-3} f(z)}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{\mathcal{Q}_\beta^{\alpha-1} f(z)}{\mathcal{Q}_\beta^\alpha f(z)} \prec q_2(z), \qquad (z \in U). \tag{5.3}$$

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